A two-stage feasible direction algorithm including variable metric techniques for non-linear optimization problems

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Abstract :

We present a feasible direction algorithm for the solution of the nonlinear programming problem with equality and inequality constraints. At each iteration a descent direction of an associated lagrangian function is defined. By modifying if, a feasible and descent direction is obtained. A linear search procedure assures the convergence of the method and the feasibility of the new iterate.

An extension of variable metric methods to the constrained case is developed. The inclusion of this approach in the algorithm produces superlinear convergence.

The algorithm is tested on some numerical examples. The results exhibit a behaviour comparable to that of the best methods known at present for nonlinear programming.

Résumé :

On présente un algorithme de directions réalisables pour résoudre les problèmes de programmation non linéaires avec contraintes d'égalité et d'inégalité. A chaque itération, une direction de descente pour une fonction de Lagrange associée au problème est d'abord calculée ; elle est ensuite modifiée pour obtenir une direction de descente réalisable ; enfin une recherche linéaire est effectuée pour assurer la réalisabilité de l'itéré suivant, et la convergence du processus.

Les méthodes à métrique variable pour problèmes sans contraintes sont étendues au cas avec contraintes. Incluant cette approche dans l'algorithme, on obtient un taux de convergence superlinéaire.

L'algorithme est testé sur quelques exemples numériques. Les résultats montrent un comportement comparable à celui des meilleures méthodes de programmation non linéaire.

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1. INTRODUCTION AND PRELIMINARIES

In ref. [9], we have developed a feasible direction algorithm for the solution of the problem

$$\min_x f(x) \tag{1.1}$$

$$\begin{aligned} \text{subject to} \quad & g_i(x) \le 0 \quad ; \quad i = 1,\dots,m \\ \text{and} \quad & g_i(x) = 0 \quad ; \quad i = m+1,\dots,m+p, \end{aligned}$$

where $f(x)$ and $g_i(x)$ are nonlinear real valued functions of a vector x in the n-dimensional Euclidian space R^n. We proved the global convergence of the method and showed that the numerical results obtained in the resolution of some test problems, are very satisfactory.

In this work we shall modify the mentioned algorithm and make an extension of variable metric methods for unconstrained minimization, to the constrained case. The inclusion of this approach in our algorithm will be done, in order to obtain superlinear convergence, and maintain the very good reliability of the method.

In unconstrained problems, quasi-Newton methods construct an approximation of the second derivative of the objective function. Under nonlinear constraints, it is also necessary to take into account the constraint curvature. The second derivative of a suitably defined Lagrangian function

$$L(\lambda,x) = f(x) + \sum_{i=1}^{m+p} \lambda_i g_i(x)$$

gives this information. We shall call

$$H(\lambda,x) = \nabla_{xx} L(\lambda,x) \; .$$

It will not be necessary for $H(\lambda,x)$ to have positive eigenvalues in directions normal to the constraint boundaries.

Positive definite Hessians can be obtained by considering extended Lagrangian functions, but they give numerical problems and need the use of active set strategies under inequality constraints.

However, it is possible to assume that $H(\lambda,x)$ has positive eigenvalues in the tangent subspace at the solution of the problem (sufficiency condition for a local minimum). Then, those methods which construct an approximation of $H(\lambda,x)$ restricted to the tangent subspace, have the advantage of manipulating positive definite matrices.

Extensions of quasi-Newton methods to constrained problems were proposed by different authors, in relation to several constrained optimization methods.

In references [6, 11,14] there are descriptions of projected gradient and reduced gradient algorithms for equality constrained problems including this approach.

In a kind of methods studied by Han [7, 8] for problems with equality and inequality constraints, a quadratic approximation of the objective function by quasi-Newton is included. In these works $H(\lambda,x)$ at the solution of the problem is approximated. Powell [13] modified the method, taking an approximation of the projection of $H(\lambda,x)$ into the tangent subspace.

In Biggs' recursive quadratic programming [1] and Tapia's diagonalized multipliers methods [5], update rules for the Lagrange multipliers are used, which depend on an approximation matrix. In order to have positive definiteness, Tapia works with the Hessian of an augmented Lagrangian function and Biggs takes the projected Hessian of the simple Lagrangian.

The algorithm presented in réf. [9] , can be considered as an iterative method for the solution of the nonlinear system of equations

$$\delta(x) = \nabla f(x) + \sum_{i=1}^{m+p} \lambda_{0i}(x) \, \nabla g_i(x) = 0 \tag{1.2}$$

where $\lambda_{0i}(x)$ are determined by an updating rule given by the method.

Quasi-Newton methods for the solution of (1.2) [5] make iterations of the kind

$$x^{k+1} = x^k - t^k (B^k)^{-1} \delta(x) \tag{1.3}$$

where B^k is an approximation of the Jacobian matrix of $\delta(x)$, noted $J(x)$, and t^k represents a step-length.

In this work the iterative scheme is similar to (1.3). We develop a technique which enables the use of symmetric and positive definite B^k. This is helpful, because as in variable metric methods for unconstrained optimization, there are many ways of updating B^k so that the algorithm is invariant under linear transformations of the variables.

In section 2 we state the algorithm, and its global convergence is proved in section 3. In section 4 we give some results concerning the Jacobian of $\delta(x)$. The variable metric approach is introduced in section 5 and superlinear convergence is studied. In section 6 we give some details concerning the practical application of the algorithm and the results obtained in the resolution of some problems.

NOTATIONS

All vector spaces are finite dimensional, the space of all $n \times m$ matrices is denoted by $R^{n\times m}$ and the transpose of M by M^T. If ϕ is a real valued function in R^n, then

$$\nabla\phi(x) \equiv \left(\frac{\partial\phi(x)}{\partial x_1}, \frac{\partial\phi(x)}{\partial x_2}, \ldots \frac{\partial\phi(x)}{\partial x_n}\right)^T .$$

We call Ω the feasible region for the inequality constraints, that is

$$\Omega \equiv \{x \in R^n; g_i \leq 0 \quad , \quad i = 1,\ldots,m\} ,$$

and denote by

$$g(x) \equiv [g_1(x), g_2(x), \ldots, g_{m+p}(x)]^T$$

and

$$A(x) \equiv [\nabla_1 g(x), \nabla g_2(x), \ldots, \nabla g_{m+p}(x)] .$$

DEFINITIONS

Definition 1.1 : A point $\bar{x}$ is a "stationary point" of problem (1.1) if there exists a vector $\bar{\lambda}$ in R^{m+p} such that the following requirements are simultaneously satisfied :

$$g_i(\bar{x}) \le 0 \quad ; \quad i = 1,\ldots,m$$

$$g_i(\bar{x}) = 0 \quad ; \quad i = m+1,\ldots,m+p$$

$$\bar{\lambda}_i g_i(\bar{x}) = 0 \quad ; \quad i = 1,\ldots,m$$

and

$$\nabla f(x) + \sum_{i=1,m+p} \bar{\lambda}_i \nabla g_i(\bar{x}) = 0 .$$

Definition 1.2 : A "Kuhn-Tucker point" of the problem (1.1) is a stationary point associated to a vector $\bar{\lambda}$ verifying

$$\bar{\lambda}_i \ge 0 \quad ; \quad i = 1,\ldots,m$$

Definition 1.3 : $d \in R^n$ is a "descent direction" of a real continuously differentiable function $\phi(x)$ in R^n, if

$$d^T \nabla\phi(x) < 0 .$$

Definition 1.4 : $d \in R^n$ is a "feasible direction" [10] of problem (1.1) at $x \in \Omega$ if for some $\tau > 0$ we have

$$x + td \in \Omega \quad \text{for all } t \in [0,\tau].$$

Definition 1.5 : A point $\bar{x}$ is a "regular point" of the problem (1.1) if the elements of the set of vectors composed by all

$$\nabla g_i(\bar{x}) \quad ; \quad i = 1,\ldots,m \quad \text{so that} \quad g_i(\bar{x}) = 0$$

and all

$$\nabla g_i(\bar{x}) \quad ; \quad i = m+1,\ldots,m+p$$

are linearly independent.

2. STATEMENT OF THE ALGORITHM

From now on we shall consider the inequality constrained problem

$$\min f(x) \tag{2.1}$$

$$\text{subject to } g_i(x) \leq 0 \quad ; \quad i = 1,\ldots,m \tag{2.2}$$

The results can be extended to the general nonlinear programming problem (1.1) applying the theory given in [9] , section 4. In réf. [9] our major objective was the development of a reliable method, having global convergence with the weakest hypothesis and with a great freedom in the selection of the different parameters defining the algorithm.

When we take also care of the final behaviour it is necessary to include second order assumptions on the problem, then we shall suppose that

i) There exists a real number a so that the region

$$\Omega_a \equiv \{x \in \Omega;\ f(x) \leq a\}$$

is compact and with nonempty interior.

ii) f and g_i are C^2 in Ω_a.

iii) All $x \in \Omega_a$ are regular points of problem (2.1).

iv) Let x^* be a Kuhn-Tucker point of problem (2.1), $\lambda^* \in R^m$ the corresponding vector of Lagrange multipliers, and T the tangent subspace to the active constraints at x^*, that is ,

$$T \equiv \{y \in R^n \ ;\ \nabla g_i(x^*)y = 0 \text{ for all } i \in I^*\},$$

where

$$I^* \equiv \{i \in (1,\ldots,m) \ ;\ g_i(x^*) = 0\} \ .$$

Then, we assume that

$$H(\lambda^*,x^*) = \nabla^2 f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla^2 g_i(x^*)$$

is positive definite on T. That is, there exists $\theta > 0$ such that

$$y^T H(\lambda^*,x^*) y \geq \theta \, y^T y \tag{2.3}$$

for all $y \in T$.

We shall define a two-stage feasible direction algorithm for the solution of problem (2.1), which is an extension of the method presented in [9].

The algorithm constructs a sequence of points $\{x^k\}$ verifying

$$g_i(x^k) < 0 \quad ; \quad i = 1,\dots,m \text{ and } k = 0,1,2,\dots;$$

and converging to a Kuhn-Tucker point of the problem. At each iteration a vector of Lagrange multipliers $\lambda_0(x^k) \in R^m$ and a direction $d_0 \in R^n$ are defined, where d_0 is a descent direction in x of

$$L(\lambda_0(x^k),x) .$$

In a second stage d_0 is modified, obtaining $d \in R^n$ which is a descent direction of the same function and a feasible direction of the problem.

In the present algorithm d_0 and d depend on a matrix B^k, we shall see different ways of defining B^k in order to have linear or superlinear local convergence.

A linear search procedure assures the global convergence of the method and the strict feasibility of all the iterates. We use an Armijo type procedure in order to get a satisfactory decrease of the Lagrangian function $L(\lambda_0^k,x)$, maintaining the feasibility of the new iterate.

The algorithm

Let

- B^k a sequence of symmetric and positive definite matrices in $R^{n\times n}$ verifying

$$\beta_1\, y^T y \le y^T B^k y \le \beta_2\, y^T y \; ; \qquad k = 0,1,2,\ldots \tag{2.4}$$

for any $y \in R^n$ and $\beta_2 > \beta_1 > 0$.

- r^k a sequence of vectors in R^m verifying

$$\beta_3 \le r_i^k \le \beta_4 \tag{2.5}$$

for $\beta_4 > \beta_3 > 0$. There is an updating rule for r_i^k, such that

$$\lim_{k\to\infty} r_i^k \lambda_{0i}^k = 1 \quad ; \quad i \in I^* \tag{2.6}$$

and

$$\lim_{k\to\infty} r_i^k = r_m \quad ; \quad i \in \bar{I}^* ,$$

where $r_m > 0$ and λ_{0i}^k will be defined later.

- $\gamma_0^k \in (0,1)$ such that $\lim_{k\to\infty} \gamma_0^k = 0$.

- $\rho_0 > 0$, $\alpha \in (0,1)$ and $c \in (0,1)$.

Step 0. - Select a strictly feasible initial point $x^0 \in \Omega_a$ and the values of ρ_0, α, and c.
Set $\rho = \rho_0$.

Step 1. - Compute $\lambda_0 \in R^m$ and $d_0 \in R^n$ by solving the linear system of equations

$$d_0 = - B^{-1}[\nabla f(x) + \sum_{i=1}^{m} \lambda_{0i} \nabla g_i(x)] \tag{2.7}$$

$$d_0^T \nabla g_i(x) = - r_i \lambda_{0i} g_i(x) \quad ; \quad i = 1,\ldots,m \tag{2.8}$$

where it is understood that $x = x^k$ and $B = B^k$.

If $d_0 = 0$, stop.

Step 2. - Compute $\rho_1 = \dfrac{(1-\alpha) d_0^T \nabla_x L(\lambda_0,x)}{|d_0|^2 \lambda_0^T RG(A^T B^{-1} A-RG)^{-1} e}$ (2.9)

where $R \equiv \text{diag}(r)$ and $G \equiv \text{diag}(g)$.

If $0 < \rho_1 < \rho$, set $\rho = \rho_1/2$.

Compute $\lambda \in R^m$ and $d \in R^n$ by solving the linear system of equations

$$d = - B^{-1}[\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x)] \tag{2.10}$$

$$d^T \nabla g_i(x) = -[r_i \lambda_i g_i(x) + \rho |d_0|^2] \quad ; \quad i = 1,\ldots,m. \tag{2.11}$$

Step 3. - For each constraint, set $\gamma_i = \gamma_0$ if $\lambda_i \geq 0$ or $\gamma_i = 1$ if $\lambda_i < 0$.

Let τ_i be the largest t_i such that

$$g_i(x + t_i d) \leq \gamma_i g_i(x) \quad ; \quad i = 1,\ldots,m \tag{2.12}$$

and set $\tau = \min \{\tau_i\}$.

Find $\bar{t} \in (0,\tau]$, the first value of the sequence

$$\{1, 1/\nu, 1/\nu^2, 1/\nu^3,\ldots\} \text{ with } \nu > 1, \tag{2.13}$$

such that

$$L(\lambda_0,x+\bar{t}d) \leq L(\lambda_0,x) + \bar{t}c[\nabla_x L(\lambda_0,x)]^T d. \tag{2.14}$$

Step 4. - Set the new iterate

$$\bar{x} = x + \bar{t}d \ .$$

Step 5. - Go to step 1.

Introducing the matrix notation given before in (2.7) to (2.11) we get

$$\lambda_0 = - (A^T B^{-1} A - RG)^{-1} A^T B^{-1} \nabla f \ , \tag{2.15}$$

$$d_0 = - B^{-1} [I - A(A^T B^{-1} A - RG)^{-1} A^T B^{-1}] \nabla f \tag{2.16}$$

$$\lambda = (A^T B^{-1} A - RG)^{-1} (-A^T B^{-1} \nabla f + \rho \ |d_0|^2 \ e) \tag{2.17}$$

and

$$d = d_0 - \rho \ |d_0|^2 \ B^{-1} A (A^T B^{-1} A - RG)^{-1} \ e \tag{2.18}$$

where

$$e \equiv \underbrace{(1,1,\dots,1)^T}_{m} \ .$$

This algorithm preserves the most important features of the method given in ref [9], in order to have a similar global behaviour.

Conditions (2.8) and (2.11), which perform a selection of the active constraints and determine the speed of convergence in directions normal to the constraint boundaries, are not modified by the inclusion of B^k.

At the stationary points of the problem, λ and λ_0 coincide with the Lagrange multipliers, according to definition 1.1 and d and d_0 are zero.

The feasible direction d is obtained by a modification of the descent direction d_0, in a similar way as in [9].

The linear search procedure maintains a monotone decrease in x of the Lagrangian function $L(\lambda_0^k, x)$. It also guarantees the feasibility of all the iterates.

3. GLOBAL CONVERGENCE THEOREM

The global convergence of the algorithm can be proved, by showing successively that :

i) d_0 computed at x^k, is a descent direction of the Lagrangian function

$$L\,(\lambda_0^k, x)$$

ii) d computed at x^k, is a descent direction of the same function, and also a feasible direction of the problem.

iii) The algorithm produces a sequence of points converging to a stationary point of the problem.

iv) If a stationary point is a point of convergence of the algorithm, it must verify Kuhn-Tucker conditions.

Because B is positive definite, we can suppose that $B^{1/2}$ exists. Then, (2.16) can be written

$$d_0 = -B^{-1/2}\,[I - B^{-1/2}\,A(A^T B^{-1} A - RG)^{-1}\,A^T B^{-1/2}]\,B^{-1/2}\,\nabla f \;. \quad (3.1)$$

Lemma 3.1 in ref. [9] can be easily extended in order to prove that

$$W = A^T\,B^{-1}\,A - RG \quad (3.2)$$

is positive definite.

Even if it is not a requisite in the proof of global convergence, it is interesting to remark that d_0 is a descent direction of the objective function. This can be shown in a similar way as in the proof of lemma 3.3 in [9] .

Because B^{-1} is positive definite, premultiplying both sides of (2.7) by $\nabla_x\, L\,(\lambda_0^k, x^k)$, we deduce that d_0 is a descent direction of $L(\lambda_0^k, x)$.

Lemma 3.1. The direction $d(x^k)$ defined in step 2 of the algorithm is a descent direction for

$$L\,(\lambda_0^k, x)\ .$$

Proof : In consequence of expression (2.18) for d, we have

$$d^T \nabla_x L = d_0^T \nabla_x L - \rho\ |d_0|^2 [\nabla_x L]^T B^{-1} A\, W^{-1}\ e \tag{3.3}$$

where all the functions are evaluated at x^k.

Considering

$$\nabla_x L = \nabla f + A\ \lambda_0\ ,$$

and expression (2.15) for λ_0, we get

$$d^T \nabla_x L = d_0^T \nabla_x L - \rho\ |d_0|^2\ \lambda_0^T\ RGW^{-1}\ e\,. \tag{3.4}$$

If

$$\lambda_0^T\ RGW^{-1}\ e \geq 0 \tag{3.5}$$

we get

$$d^T \nabla_x L \leq d_0^T \nabla_x L \leq 0$$

and the lemma is proved. Suppose now that

$$\lambda_0^T\ RGW^{-1} e < 0\ ,$$

considering the definition of ρ in step 2 of the algorithm we have

$$d^T \nabla_x L \leq \alpha\ d_0^T\ \nabla_x L \tag{3.6}$$

which implies that d is a descent direction of the Lagrangian function. □

Note that inequality (3.6) gives an upperbound of the derivative of $L(\lambda_0^k, x)$ in the direction d, in terms of its derivative in the direction d_0.

Since all x^k are in Ω_a compact and f, g_i are continuously differentiable, ∇f and ∇g_i are bounded in Ω_a.

As W is positive definite in Ω_a, W^{-1} is also bounded in Ω_a. It follows from (2.15) that there exists $\beta_5 < \infty$ such that

$$- \lambda_0^T RGW^{-1} e \leq \beta_5 \tag{3.7}$$

for all $x \in \Omega_a$.

When (2.9) produces a positive ρ_1, conditions (2.4) and (3.7) give

$$\rho_1 \geq \frac{(1-\alpha)\beta_2^{-1}}{\beta_1^{-2} \; \beta_5} \tag{3.8}$$

Then, ρ defined in step 2 of the algorithm remains constant after a finite number of iterations.

We are now in a position to show the existence of $\beta > 0$ such that

$$|d_0|^2 \geq \beta \; |d|^2 \tag{3.9}$$

holds in Ω_a. The proof can be developed like in ref. [9]; corollary 3.1.

<u>Lemma 3.2</u>. There is a real positive number τ^{max} such that d is a feasible direction of problem (2.1), according to the definition 1.4, for all $\tau \in (0, \tau^{max}]$ and all $x \in \Omega_a$.

The proof is given in ref. [9] lemma 3.5.

Comparing the present algorithm with the one given in ref. [9] we remark that $\lambda_0(x)$ and $\lambda(x)$ are no longer continuous in x. This is due to the introduction of r^k and B^k, which can change at each iteration.

However, this fact does not produce major difficulties, since the values assumed by λ_0 and λ at the stationary points of the problem are independent of r^k and B^k.

Lemma 3.3. Any accumulative point x^* of the sequence $\{x^k\}$ generated by the algorithm is a stationary point of the problem.

Proof : If the sequence terminates

$$d = 0$$

and (3.6) gives

$$d_0^T \nabla_x L \geq 0 ,$$

or

$$-[\nabla_x L]^T B^{-1} \nabla_x L \geq 0 .$$

Since B^{-1} is positive definite, it follows that

$$\nabla_x L = 0$$

and (2.7), (2.8) prove the result.

Suppose now that the sequence

$$\{x^k\} \to x^* \quad ; \quad k \in K$$

Because of condition (2.4) on B^k, there exists a sequence $K_1 \subset K$ such that

$$\lambda_0^k \to \lambda_0^* \text{ and } B^k \to B^* \quad ; \quad k \in K_1 ,$$

with B^* positive definite.

Consider the step-length t^k defined in step 3 of the algorithm, there are two possibilities

a) there is a sequence $K_2 \subset K_1$ such that

$$t^k \to t^* \neq 0 \quad ; \quad k \in K_2$$

b) there is a sequence $K_3 \subset K_1$ such that

$$t^k \to 0 \quad ; \quad k \in K_3$$

Condition (2.14) can be written

$$L(\lambda_0^k, x^{k+1}) \leq L(\lambda_0^k, x^k) + t^k c[\nabla_x L(\lambda_0^k, x^k)]^T d^k \tag{3.10}$$

If a) is true, taking limits for $k \to \infty$ in both sides of (3.10) we have

$$L(\lambda_0^*, x^*) \leq L(\lambda_0^*, x^*) + t^* c[\nabla_x L(\lambda_0^*, x^*)]^T d^* ,$$

where

$$d^* = \lim_{k\to\infty} d^k = d(x^*) .$$

It follows that

$$[\nabla_x L(\lambda_0^*, x^*)]^T d^* \geq 0 ,$$

and as (3.6) is also valid in the limit for $k \to \infty$, we have

$$-[\nabla_x L(\lambda_0^*, x^*)]^T B^{-1} \nabla_x L(\lambda_0^*, x^*) \geq 0 .$$

In consequence

$$\nabla_x L(\lambda_0^*, x^*) = 0$$

and the lemma is true.

If the alternative b) is verified, we can find k_0 such

$$t^k > 0 \quad \text{and} \quad \nu . t^k \le \tau_0 \quad ; \quad k > k_0 \quad , \quad k \in K_3$$

with τ_0 given in lemma 3.2.

Consider the sequence

$$t'^k = \nu . t^k \quad ; \quad k \in K_3 \quad , \quad k > k_0 \; .$$

It follows from step 3 in the algorithm that t'^k does not satisfy (2.14), then

$$L(\lambda_0^k, x^k + t'^k d^k) > L(\lambda_0^k, x^k) + t'^k \, c[\nabla_x L(\lambda_0^k, x^k)]^T d^k \; ,$$

or

$$[L(\lambda_0^k, x^k + t'^k d^k) - L(\lambda_0^k, x^k)]/t'^k > c\,[\nabla_x L(\lambda_0^k, x^k)]^T d^k \; .$$

We can find $\theta^k \in [0,1]$ so that

$$[L(\lambda_0^k, x^k + t'^k d^k) - L(\lambda_0^k, x^k)]/t'^k = [\nabla_x L(\lambda_0^k, x^k + \theta^k t'^k d^k)]^T d^k \; ,$$

and in consequence

$$\nabla_x [L(\lambda_0^k, x^k + \theta^k t'^k d^k)]^T d^k > c\,[\nabla_x L(\lambda_0^k, x^k)]^T d^k \; .$$

Passing to the limit,

$$[\nabla_x L(\lambda_0^*, x^*)]^T d^* \ge c\,[\nabla_x L(\lambda_0^*, x^*)]^T d^* \; . \tag{3.11}$$

On the other hand, (3.6) implies that

$$[\nabla_x L^T(\lambda_0^*, x^*)]^T d^* \le 0 \; . \tag{3.12}$$

In consequence of (3.11) and (3.12), we have

$$[\nabla_x L\ (\lambda_0^*, x^*)]^T\ d = 0\ ,$$

since $c < 1$. Then, the result of the lemma is obtained in the same way as in possibility a). □

The proof of global convergence of the method is completed by the following theorem.

Theorem 3.1. Any accumulation point of any sequence generated by the algorithm is a Kuhn-Tucker point of the problem.

The proof is given in ref. [9] theorem 3.1.

4. SOME RESULTS CONCERNING THE JACOBIAN MATRIX

The first step of the algorithm of section 2, can be considered as an iterative method for the solution of the nonlinear system of equations

$$\delta(x) \equiv \nabla f(x) + \sum_{i=1}^{m} \lambda_{0i}(x)\, \nabla g_i(x) = 0$$

with iterates of the form (1.3) and $\lambda_0(x)$ given by the updating rule (2.15). This interpretation shows the great importance of $\mathcal{J}(x)$, the Jacobian matrix of $\delta(x)$, in the study of the local convergence of the method.

In this section we shall suppose that B and r_i are fixed during the iterative process, r_i verifying

$$r_i = 1/\lambda_{0i}(x^*) \quad , \quad \text{for all} \quad i \in I^* ,$$

and

$$r_i = r_m \quad , \quad \text{for all} \quad i \in \bar{I}^* .$$

Under this assumptions, we shall give some results concerning $\mathcal{J}(x^*)$, which will depend of the way of defining B.

Consider the tangent subspace $\mathcal{T}$, defined in section 2, and call $\mathcal{N}$ the subspace normal to $\mathcal{T}$.

Let

$$A^*(x^*) = [\nabla g_i(x^*)]_{i \in I^*} ,$$

then

$$N = A^*(x^*)[A^{*T}(x^*)A^*(x^*)]^{-1} A^{*T}(x^*) \tag{4.4}$$

is the projection matrix on $\mathcal{N}$, and

$$T = I - N \tag{4.5}$$

the projection matrix on $\mathcal{T}$.

In order to have good convergence characteristics, it is necessary to find the conditions under which $J(x^*)$ is positive definite in $\mathcal{T}$. These conditions will be given in Lemma (4.1) and Theorem (4.1). Furthermore, under slightly stronger assumptions on B, we shall prove that $J(x^*)$ is positive definite in R^n.

Notice that, when $k \to \infty$, $\lambda_i(x^*)$ of the non active constraints and its derivatives depend, in particular, on r_m.

Lemma 4.1. Given $\theta_1 > 0$, there exists $\bar{r} > 0$ such that for all $i \in \bar{I}^*$, and all $r_m \geq \bar{r}$ we have

$$|\nabla \lambda_{0i}(x^*)| \leq \theta_1 \ . \tag{4.6}$$

Proof : (2.15) is equivalent to

$$(A^T B^{-1} A - RG)\lambda_0 = - A^T B^{-1} \nabla f \quad , \tag{4.7}$$

where $A, G, \nabla f$, and λ_0 are functions of x. Derivating both members with respect to x_j we have

$$(A^T B^{-1} A - RG) \frac{\partial \lambda_0}{\partial x_j} + [\frac{\partial}{\partial x_j}(A^T B^{-1} A - RG)]\lambda_0 = - \frac{\partial}{\partial x_j}(A^T B^{-1} \nabla f)$$

and

$$\frac{\partial \lambda_0}{\partial x_j} = - (A^T B^{-1} A - RG)^{-1} b_j \ , \tag{4.8}$$

where

$$b_j \equiv \frac{\partial}{\partial x_j} (A^T B^{-1} \nabla f) + [\frac{\partial}{\partial x_j}(A^T B^{-1} A - RG)]\lambda_0 \ , \tag{4.9}$$

$b_j \in R^m$.

Since

$$\lambda_{0i}(x^*) = 0 \quad ; \quad i \in \bar{I}^* ,$$

and

$$\frac{\partial(RG)}{\partial x_j}$$

is diagonal, we deduce that $b_j(x^*)$ does not depend on r_i, for $i \in \bar{I}^*$.

Let us consider the linear system of equations

$$\frac{\partial \lambda_0(x^*)}{\partial x_j} = - (A^T B^{-1} A - RG)^{-1} b_j \Big|_{x=x^*} \tag{4.10}$$

and apply Cramer's rule for its' resolution. Then

$$\frac{\partial \lambda_{0i}(x^*)}{\partial x_j} = \frac{\det(W^i)}{\det(W)} \Big| _{x=x^*} ,$$

where W was defined in (3.2) and W^i is obtained substituting by $b_i(x^*)$ the i-th row of W.

Consider now W, then

$$W_{ii}(x^*) = (\nabla g_i)^T B^{-1} \nabla g_i |_{x=x^*} \quad ; \quad i \in I^* ,$$

and
$$W_{ii}(x^*) = [(\nabla g_i)^T B^{-1} \nabla g_i - r_i g_i]|_{x=x^*} \quad ; \quad i \in \bar{I}^* .$$

Let us call $q = \text{Card } \bar{I}^*$, and take $r_i = r_m$ for all $i \in \bar{I}^*$. Then $\det[W(x^*)]$ is a q-th degree polynomial in r_m, the highest order term being

$$(-1)^q \, r_m^q \, [\prod_{i \in \bar{I}^*} g_i(x^*)] \det[A^{*T}(x^*) \, B^{-1} \, A^*(x^*)] \quad ,$$

which is not zero because of the regularity condition. Since $b_j(x^*)$ is independent of r_m, $\det[W^i(x^*)]$ is a polynomial in r_m of degree less than or equal $(q-1)$, for all $i \in \bar{I}^*$.

In consequence

$$\lim_{r_m \to \infty} \left|\frac{\partial\lambda_{0i}(x^*)}{\partial x_j}\right| = 0 \quad ; \quad i \in \bar{I}^* ,$$

so there exists $\bar{r}$ such that for all $r_m \geq \bar{r}$

$$\left|\frac{\partial\lambda_{0i}(x^*)}{\partial x_j}\right| \leq \frac{\theta_1}{n^{1/2}} \quad ; \quad i \in \bar{I}^* . \tag{4.11}$$

Inequality (4.11) gives the result of the lemma. □

Theorem 4.1. If $H(x^*)$ verifies condition iv) in section 2, there exists $\bar{r} > 0$ such that for all $v \in R^n$, and all $r_m \geq \bar{r}$

$$v^T \, T J(x^*) \, T \, v \geq \frac{1}{2} \, \theta |T \, v|^2 \tag{4.12}$$

Proof : It follows from (1.2) that

$$J(x) = \nabla^2 f(x) + \sum_{i=1}^{m} \lambda_{0i}(x) \, \nabla^2 g_i(x) + \sum_{i=1}^{m} \nabla g_i(x)[\nabla\lambda_{0i}(x)]^T \tag{4.13}$$

or

$$J(x) = H(x) + \sum_{i=1}^{m} \nabla g_i(x)[\nabla\lambda_{0i}(x)]^T . \tag{4.14}$$

Since at a Kuhn-Tucker point

$$T \nabla g_i(x^*) = 0 \quad ; \quad i \in I^*,$$

we have

$$v^T \, TJ(x^*)T \, v = v^T \, T \, H(x^*) \, T \, v + v^T \, T \, \{ \sum_{i \in \bar{I}^*} [\nabla g_i(x^*)[\nabla\lambda_{0i}(x^*)]^T\} \, Tv|$$

and as

$$v^T \, T \, H(x^*) T \, v \geq 0 \tag{4.15}$$

$$v^T \, T \, J(x^*) \, T \, v \geq v^T \, T \, H(x^*) \, T \, v - |v^T \, T \, \{ \sum_{i \in \bar{I}^*} \nabla g_i(x^*)[\nabla\lambda_{0i}(x^*)]^T\} Tv$$

(4.15)

is true.

Since

$$|v^T\, T\{ \sum_{i\in\bar{I}^*} \nabla g_i(x^*)[\nabla\lambda_{0i}(x^*)]^T\} Tv| \leq |T\,v|^2 \sum_{i\in\bar{I}^*} |\nabla\lambda_{0i}(x^*)|\,|\nabla g_i(x^*)| \quad ,$$

taking in Lemma 4.1.

$$\theta_1 = \theta/(\,2\sum_{i\in\bar{I}^*} |\nabla g_i(x^*)|\,)\ ,$$

we have

$$|v^T\, T\{ \sum_{i\in\bar{I}^*} \nabla g_i(x^*)[\nabla\lambda_{0i}(x^*)]^T\}\, Tv| \leq \frac{1}{2}\,\theta\, |T\,v|^2\ .$$

Then, the assumption iv) in section 2 gives (4.12). □

Consider now the updating rule for λ_{0i}, given in (2.8). Derivating, we get an expression for $J(x)$

$$J^T(x)\ B^{-1}\ \nabla g_i(x) + \nabla^2 g_i(x) B^{-1}\delta(x) = r_i[\lambda_{0i}(x)\nabla g_i(x) + g_i(x)\nabla\lambda_{0i}(x)]\ ;$$

$$i = 1,\ldots, m. \tag{4.16}$$

In a Kuhn-Tucker point of the problem, we have

$$\delta(x^*) = 0\ ,$$

$$g_i(x^*) = 0 \quad ; \quad i \in I^*,$$

and

$$\lambda_{0i}(x^*) = 0 \quad ; \quad i \in \bar{I}^*.$$

Then, for $i \in I^*$, we have

$$J^T(x^*)\ B^{-1}\ \nabla g_i(x^*) = \nabla g_i(x^*) \tag{4.17}$$

and for $i \in \bar{I}^*$

$$J^T(x^*)\ B^{-1}\ \nabla g_i(x^*) = r_m\ g_i(x^*)\ \nabla\lambda_{0i}(x^*) \quad , \tag{4.18}$$

which give m conditions on $J(x^*)$.

In particular, equations (4.17) show that

$$\nabla g_i(x^*) \quad , \quad \text{for all } i \in I^* \ ,$$

are eigenvectors of the matrix $J^T(x^*)\ B^{-1}$ and the corresponding eigenvalues are unitary.

In consequence, for any $v \in R^n$

$$J^T(x^*)\ B^{-1}\ N\ v = N\ v\ . \tag{4.19}$$

<u>Lemma 4.2</u>. $J(x^*)$ is not singular.

<u>Proof</u> : We shall show that, if $v \in R^n$,

$$J(x^*)\ v = 0 \tag{4.20}$$

implies $v = 0$.

Considering

$$v = T\ v + N\ v\ ,$$

(4.20) gives

$$J\ (x^*)\ T\ v = -\ J(x^*)\ N\ v\ , \tag{4.21}$$

and

$$v^T\ N\ B^{-1}\ J(x^*)\ T\ v = -\ v^T\ N\ B^{-1}\ J(x^*)\ N\ v\ .$$

Then, by (4.19), we have

$$v^T N T v = - v^T N v ,$$

which implies

$$N v = 0 .$$

Substitution in (4.21) gives

$$J(x^*) T v = 0 ,$$

and by Theorem 4.1

$$T v = 0 .$$

In consequence

$$v = 0 . \qquad \square$$

It is clear that the division of the Euclidian space in T and N is very important in the study of local convergence of the algorithm. Conditions (2.8) on d_0, which impose a speed of approximation to the active constraints, are concerned with the local convergence on N. On the other side, (2.7) gives the local convergence on T, which will depend on the way of defining B.

It seems then reasonable to maintain this separation between T and N in the definition of B. Thus, from now on we shall suppose that N is invariant under B. In other words, if $v \in N$, then $Bv \in N$.

Since B is symmetric, T is also invariant under B and there are two complementary sets of eigenvalues of B, $\{u_{ni}\}$ spanning N and $\{u_{ti}\}$ spanning T.

Under this assumption, we shall prove

Lemma 4.3. $J(x^*)$ and B have the same eigenvectors and eigenvalues in N.

Proof : Call μ_{ni} the eigenvalue of B corresponding to u_{ni}. It follows from (4.19) that

$$J^T(x^*)\ B^{-1}\ u_{ni} = u_{ni}\ ,$$

then

$$J^T(x^*)\ \mu_{ni}^{-1}\ u_{ni} = u_{ni}\ ,$$

or

$$J(x^*)\ u_{ni} = \mu_{ni}\ u_{ni}$$

which gives the result. □

Lemma 4.4. T is invariant under $J(x^*)$.

Proof : Let us consider $v \in T$, it follows from (4.19) that

$$v^T\ J^T(x^*)\ B^{-1}\ u_{ni} = v^T\ u_{ni} \quad \text{for all} \quad ni\ .$$

In consequence

$$v^T\ J^T(x^*)\ u_{ni} = 0 \quad \text{for all} \quad ni\ ,$$

which implies

$$J(x^*)\ v \in T,$$

since $\{u_{ni}\}$ are a basis of N . □

Theorem 4.2. $J(x^*)$ is positive definite.

Proof : Consider $v \in R^n$, it follows from Lemma 4.3, that

$$v^T\ T\ J(x^*)\ N\ v = 0\ , \tag{4.21}$$

and

$$v^T N J(x^*) N v > 0 \tag{4.22}$$

Lemma 4.4 gives

$$v^T N J(x^*) T v = 0 . \tag{4.23}$$

Then,

$$v^T J(x^*) v = v^T N J(x^*) N v + v^T T J(x^*) T v ,$$

and considering (4.22) and Theorem 4.1, we get the results of the present theorem. □

Let us sum up; when N is invariant under B, we have :

i) $J(x^*)$ coincides with B in N.

ii) N and T are also invariant under $J(x^*)$.

iii) $J(x^*)$ is positive definite in R^n.

iv) $J(x^*)$ is not necessarily symmetric in T . A nonsymmetric perturbation on the Hessian of the Lagrangian, can be introduced by the inactive constraints, as is obvious from (4.14). This perturbation goes to zero as r_m increases.

v) It follows that $J(x^*)$ can be considered as a perturbation of a symmetric and positive definite matrix in R^n.

We shall not present further results concerning the local convergence of the algorithm, with B defined in this section. However, it seems possible to show linear convergence, and obtain results similar to those given by Cohen [3] in relation to descent methods for minimizing real-valued functions in R^n.

5. AN EXTENSION OF VARIABLE METRIC METHODS TO CONSTRAINED OPTIMIZATION PROBLEMS

In this section we shall consider B and give some additional conditions on it, in order to obtain superlinear convergence.

We shall define a class of symmetric and positive definite matrices, and prove that if B is near a matrix of this family, then B must also be near $\mathcal{J}(x)$.

Consider

$$\mathcal{Y}_{C,r_m}(x) = \mathcal{H}(x) + \sum_{i\in\bar{I}^*} \frac{\nabla g_i(x)[\nabla g_i(x)]^T}{r_m\, g_i(x)} + C\, A(x)\, \Lambda_0(x) A^T(x)\ , \tag{5.1}$$

where $C > 0$ and $\Lambda_0(x) = \mathrm{diag}[\lambda_0(x)]$.

The positive definiteness of $\mathcal{Y}_{C,r_m}(x^*)$ is assured for C and r_m sufficiently large by the two following lemmas :

Lemma 5.1. There is a $\bar{r} > 0$ such that, for all $r_m \geq \bar{r}$

$$v^T\, T\, \mathcal{Y}(x^*)\, T\, v \geq \frac{1}{2}\theta\, |T\, v|^2$$

for any $v \in R^n$, and θ given in assumption iv), section 2.

Proof : Since

$$T\, A(x^*)\, \Lambda_0(x^*)\, A^T(x^*)\, T = 0\ , \tag{5.2}$$

and considering (2.3), we have

$$v^T\, T\, \mathcal{Y}_{C,r_m}(x^*)\, T\, v \geq \theta\, |T\, v|^2 - \frac{1}{r_m} v^T\, T\, M\, T\, v\ ,$$

where

$$M = \sum_{i \in \bar{I}^*} \frac{\nabla g_i(x^*) [\nabla g_i(x^*)]^T}{- g_i(x^*)}$$

is positive definite.

If we call μ_M the highest eigenvalue of M, we can deduce that

$$\bar{r} = 2 \, \mu_M / \theta$$

verifies the requirements of the lemma. □

Lemma 5.2. For any $r_m \geq \bar{r}$ defined above, there is a $\bar{C} > 0$ such that, for all $C \geq \bar{C}$, $\mathcal{V}_{C,r_m}(x^*)$ is positive definite.

The proof is a consequence of lemma 5.1.

In the following, we shall consider $\mathcal{V}(x) = \mathcal{V}_{C,r_m}(x)$ with C and r_m given by lemmas 5.1 and 5.2.

Our aim is to show that, if

$$|x - x^*|$$

and

$$\frac{|T[\mathcal{V}(x) - B] \, T \, d(x)|}{|d(x)|}$$

are small enough, then

$$\frac{|F(x) - x^*|}{|x - x^*|}$$

is as small as we want, where $F(x) = x + d(x)$.

We call

$$\mathcal{B} \equiv \{B \in R^{n\times n} \text{ symmetric, positive definite, and } \mathcal{N} \text{ invariant under } B \; ; \; \|B^{-1}\| < \beta_1 \; , \; \|B\| < \beta_2\}$$

where $\beta_2 > \beta_1 > 0$ and $\|.\|$ is any matricial norm, consistent with the ℓ_2 vector norm $|v| = v^T v$.

Lemma 5.3. Given $\varepsilon_1 > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that whenever $x \in \Omega_a$ and $B \in \mathcal{B}$ satisfy

$$|x - x^*| < \delta_1 \tag{5.3}$$

and

$$\frac{|T[\mathcal{Y}(x) - B]\, T\, d(x)|}{|d(x)|} < \delta_2 \, , \tag{5.4}$$

then

$$\frac{|[\mathcal{J}(x) - B]\, d(x)|}{|d(x)|} < \varepsilon_1 \tag{5.5}$$

Proof : It follows from (4.14) that

$$T\, \mathcal{J}(x)\, T = T\, \mathcal{H}(x)\, T + T\{\sum_{i=1}^{m} \nabla g_i(x)[\nabla \lambda_{0i}(x)]^T\}T \tag{5.6}$$

Considering (4.16), we get

$$T\{\sum_{i=1}^{m} \nabla g_i(x)[\nabla\lambda_{0i}(x)]^T\}^T = T\, D(x)\, B^{-1}\, \mathcal{J}(x)\, T + \psi_B(x), \tag{5.7}$$

where

$$D(x) = \sum_{i\in \bar{I}^*} \frac{\nabla g_i(x)\, [\nabla g_i(x)]^T}{r_m\, g_i(x)} \, ,$$

and

$$\psi_B(x) = T\{ \sum_{i\in I^*} \nabla g_i(x)[\nabla\lambda_{0i}(x)]^T + \sum_{i\in \bar{I}^*} \frac{\nabla g_i(x)\ \delta^T(x)\ B^{-1}\ \nabla^2 g_i(x)}{r_m\ g_i(x)} -$$

$$- \sum_{i\in \bar{I}^*} \frac{\nabla g_i(x)\ [\nabla g_i(x)]^T \lambda_{0i}(x)}{g_i(x)} \}T \ ,$$

($\psi_B(x^*) \equiv 0$ for any B because of Kuhn-Tucker conditions).

It follows from (5.6) and (5.7)

$$T\ J(x)\ T = T\ H(x)\ T + T\ D(x)\ B^{-1}\ [J(x)-B]T + T\ D(x)\ T + \psi_{B1}(x) \ ,$$

where

$$\psi_{B1}(x) = \psi_B(x) + C\ T\ A(x)\ \Lambda_0(x)\ A^T(x)\ T \ .$$

Then,we get

$$T\ [B-D(x)]\ B^{-1}\ [J(x)-B]T = T[Y(x)-B]T + \psi_{B1}(x) \ ,$$

and

$$T[B-D(x)]B^{-1}\ T\ [J(x)-B]T = T[Y(x)-B]T + \psi_{B2}(x) \qquad (5.8)$$

where

$$\psi_{B2}(x) = \psi_{B1}(x) - T[B-D(x)]B^{-1}\ N[J(x)-B]T \ ,$$

($\psi_{B2}(x^*) \equiv 0$ for any B in consequence of Lemma 4.4).

Since $D(x)$ is negative semidefinite and B is positive definite, we deduce that $[B-D(x)]B^{-1}$ is not singular. Let β_6 be such that, for any $v \in R^n$,

$$|T[B-D(x)]B^{-1}\ T\ v| \le \beta_6\ |T\ v| \ .$$

It follows from (5.8) that

$$\frac{|T[J(x)-B]T\ d(x)|}{|d(x)|} \le \frac{1}{\beta_6}\frac{|T[y(x)-B]T\ d(x)|}{|d(x)|} + \frac{1}{\beta_6}\frac{|\psi_{B2}(x)\ d(x)|}{|d(x)|}\ , \tag{5.9}$$

In consequence,

$$\frac{|[J(x)-B]d(x)|}{|d(x)|} \le \frac{1}{\beta_6}\frac{|T[y(x)-B]T\ d(x)|}{|d(x)|} + \psi_{B3}(x)\ , \tag{5.10}$$

where

$$\psi_{B3} = \frac{1}{\beta_6}\ \frac{|\psi_2(x)d(x)|}{|d(x)|} + \frac{|N[J(x)-B]T\ d(x)|}{|d(x)|} + \frac{|[J(x)-B]N\ d(x)|}{|d(x)|}\ .$$

Because of lemmas 4.3 and 4.4, we have

$$N[J(x^*)-B]T \equiv 0$$

and

$$[J(x^*)-B]N \equiv 0 \quad \text{, for any } B,$$

then

$$\lim_{x\to x^*} \psi_{B3}(x) = 0 \quad \text{for any } B.$$

Thus, there exists a constant $\delta_1 > 0$ such that if $x \in \Omega_a$ and

$$|x - x^*| < \delta_1\ ,$$

then

$$\psi_{B3}(x) < \varepsilon_1/2$$

for any $B \in \mathcal{B}$.

The result follows taking

$$\delta_2 = \frac{1}{2}\,\beta_6\ \varepsilon_1\ . \qquad \square$$

As we shall see, inequality (5.4) gives a rule for the determination of B in order to have superlinear convergence.

Clearly, if $||[\mathcal{Y}(x)-B]||$ is small enough, then (5.4) holds. However, (5.4) only requires that B approaches $\mathcal{Y}(x)$ along the tangent projection of $d(x)$, which is a weaker assumption.

This way of defining B makes possible the use of symmetric and positive definite updating matrices B, which is in concordance with the theoretical development made up to now and has a great importance in numerical applications.

In the following of this chapter, we shall suppose that the step-length given in step 3 of the algorithm is

$$\bar{t} = 1 \ .$$

<u>Lemma 5.4</u>. Given $\varepsilon_2 > 0$, there exist $\delta_3 > 0$ and $\delta_4 > 0$ such that, whenever $x \in \Omega_a$ and $B \in \mathcal{B}$ satisfy

$$|x - x^*| < \delta_3 \tag{5.11}$$

and

$$\frac{|T[\mathcal{Y}(x)-B]T\ d(x)|}{|d(x)|} < \delta_4 \ , \tag{5.12}$$

then

$$\frac{|\delta[F(x)]|}{|d(x)|} < \varepsilon_3 \ , \tag{5.13}$$

with $\delta(x)$ given in (1.2).

<u>Proof</u> : Using a second order Taylor development, we get

$$\delta[F(x)] = \delta(x) + \mathcal{J}(x)d(x) + o\ |d(x)|^2 \tag{5.14}$$

It follows from (2.7), that

$$\delta(x) = - B\, d_0(x) ,$$

and by (2.18) with $\bar{t} = 1$, we get

$$\delta(x) = - B\, d(x) + \rho\, |d_0|^2\, A\, W^{-1}\, e|_x . \tag{5.15}$$

Substitution in (5.12) and division by $|d(x)|$, gives

$$\frac{|\delta[F(x)]|}{|d(x)|} \leq \frac{|[J(x)-B]\, d(x)|}{|d(x)|} + \psi_{B4}(x) , \tag{5.16}$$

where

$$\psi_{B4}(x) = \frac{|\rho\, AW^{-1} e|\; |d_0|^2}{|d(x)|} + o\, |d(x)| .$$

Since (3.9) and as AW^{-1} is bounded, we have

$$\lim_{x \to x^*} |\rho AW^{-1} e|\; |d_0|\,\Big|_x = 0 ,$$

and

$$\lim_{x \to x^*} \psi_{B4}(x) = 0 , \quad \text{for any } B \in \mathcal{B}$$

Thus, there exists a constant $\delta_3' > 0$ such that if $x \in \Omega_a$ and

$$|x - x^*| < \delta_3' ,$$

then

$$\psi_{B4}(x) < \varepsilon_2/2$$

for any $B \in \mathcal{B}$.

Taking in lemma 5.3

$$\varepsilon_1 = \varepsilon_2/2 ,$$

we get δ_4 and $\delta_3 = \min\,(\delta_3', \delta_i)$. □

Theorem 5.1. Given $\varepsilon > 0$, there exist $\delta_4 > 0$ and $\delta_5 > 0$ such that, whenever $x \in \Omega_a$ and B satisfy

$$|x - x^*| < \delta_5 \tag{5.17}$$

and

$$\frac{|T[\mathcal{Y}(x)-B]T\ d(x)|}{|d(x)|} < \delta_4 \tag{5.18}$$

then

$$\frac{|\mathcal{F}(x)-x^*|}{|x - x^*|} < \varepsilon. \tag{5.19}$$

Proof : Since, for a given B, $\mathcal{J}(x^*)$ is not singular, there exist $c_1 > 0$ and $c_2 > 0$ such that, if

$$|x - x^*| < c_2 , \tag{5.20}$$

then

$$|\delta[\mathcal{F}(x)]| > c_1\ |\mathcal{F}(x) - x^*| . \tag{5.21}$$

Applying lemma (5.4), we get,

$$\frac{|\mathcal{F}(x) - x^*|}{|d(x)|} < \frac{\varepsilon_2}{c_1} .$$

It follows that

$$\frac{|\mathcal{F}(x) - x^*|}{|\mathcal{F}(x) - x^*| + |x - x^*|} < \frac{\varepsilon_2}{c_1} ,$$

which implies

$$\frac{|\mathcal{F}(x) - x^*|}{|x - x^*|} < \frac{\varepsilon_2}{c_1 - \varepsilon_2} .$$

Then, (5.19) is verified if

$$\varepsilon_2 < c_1\varepsilon/(1+\varepsilon) \ .$$

The value of δ_4 is obtained in lemma 5.4, and δ_5 is the smallest of δ_3, given in the same lemma, and c_2. □

Since the algorithm given in section 2 converges globally, if we state some updating rule for B^k such that, for a given $\delta_4 > 0$, (5.17) is verified after a finite number of iterations, then the rate of convergence of the method will be Q-superlinear. This is a very strong result in nonlinearly constrained optimization.

6. PRACTICAL IMPLEMENTATION AND NUMERICAL EXAMPLES

Superlinear convergence of the sequence $\{x^k\}$ defined by the algorithm depends of the method of calculating $\{B^k\}$.

Because the formulae that we use are all related to a single iteration, we drop the superscripts k. We write $\bar{x}$ instead of x^{k+1}.

The formula prefered by several authors for defining $\bar{B}$ when there are no constraints is the BFGS updating rule

$$\bar{B} = B - \frac{Bdd^T B}{d^T Bd} + \frac{\eta\,\eta^T}{d^T\,\eta} \tag{6.1}$$

where the vector η is the change of the gradient of the function. If the matrix $\bar{B}$ is positive definite, B is also positive definite provided that the inequality

$$d^T\,\eta > 0 \tag{6.2}$$

holds.

We let

$$\gamma = \nabla_x\, L(\lambda_0,\bar{x}) - \nabla_x\, L(\lambda_0,x)\ .$$

In the application of our algorithm, we cannot assure that it is possible to choose the step-lenght $\bar{t}$ so that $d^T\gamma$ is positive, even if r_m is large enough and $J(x^*)$ is positive definite.

In order to satisfy the positive definiteness condition (6.2) we shall adopt a procedure for the definition of η, given by Powell in references [12,13].

Consider the number θ given by

$$\theta = \begin{cases} 1 \ , & d^T\gamma \geq 0.2 \ d^T Bd \\ \dfrac{0.8 \ d^T \ Bd}{d^T \ Bd - d^T\gamma} \ , & d^T\gamma < 0.2 \ d^T \ Bd \ , \end{cases}$$

then η is defined to be the vector

$$\eta = \theta\gamma + (1-\theta) \ Bd \ , \tag{6.3}$$

and $\bar{B}$ is the matrix given in (6.1). Thus η is the closest vector to γ of the form (6.3) that satisfies

$$d^T\eta \geq 0.2 \ d^T \ Bd \ .$$

We state also an updating rule for v_i, in the following way :

If $r_i < 1/r_m$ take $\bar{r}_i = r_m$. Otherwise take

$$\bar{r}_i = 1/ \lambda_i \ .$$

The given algorithm was applied to several test problems. We report here our experience with four problems, described in a work by Hock at al.[10]. We shall identify them with the same number as in the mentioned work.

Problem 35 - (Beale's problem) has 3 design variables and 4 linear equality constraints.

Problem 43 - (Rosen - Suzuki, [2]) has 4 design variables and 3 nonlinear inequality constraints.

Problem 86 - (Colville N° 1, [4] has 5 design variables and 15 linear inequality constraints.

Problem 117- (Colville N° 2, [4] has 15 design variables, 5 nonlinear inequality constraints and 15 linear inequality constraints.

In all of them, the initial point is feasible for the inequality constraints. The iterative process was stopped with a value of the function correct to five significant digits and the inequalities verified.

The tests were performed on a HB-68 DPS/Multics computer. All the calculations were carried out in single precision (27 bit mantissa), except problem 117, calculated in double precision.

In Table 6.1 we give our final results and also intermediate results in which the objective function value is correct up to two significative digits.

Problem	Iterations	Func. and grad. evaluations	Function value
35	3	4	0.1130046
	6	7	0.1111178
43	6	8	-43.85651
	9	11	-43.99907
86	5	5	-32.14405
	9	9	-32.34860
117	35	37	32.93569
	48	50	32.34877

Table 6.1.

We shall compare our results with those obtained in a work by Hock et al. [10], and with our results in ref. [9]. In [10] the best performances are given by VFØ2A and OPRQP programs.

In the numerical tests shown in [10], in general VFØ2AD needed a less number of functions and gradient evaluations than OPRQP but in counterpart, OPRQP used less calculation time.

In the three first examples, the number of evaluations with the present method is similar to the number with VFØ2AD and smaller than in the results obtained in ref. [9] . In problem 117 we needed a number of evaluations much greater than with VFØ2AD. This may be due to a nonconvex feasible region.

Even if we have no computation time informations, we estimate that the calculation needed by our method is similar to that of OPRQP program and significally smaller that VFØ2AD.

Considering also that the present is a feasible method, we conclude that the numerical results are very satisfactory.

The method proved also to be very reliable. This is due to the fact that active set strategies are unnecessary, and that the linear search scheme doesn't introduce discontinuities.

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